USING LSE METHOD

- From the model , make the error term the subject

$y = \alpha + \beta x + \in, \qquad make\ the\ error\ term\ the\ subject : \in = y - \alpha - \beta x$

- Form sums of squares

$$\sum \in^2 = \sum (y - \alpha - \beta x)^2$$

- Minimize the sums of squares (differentiate with respect to the parameters) and equate the results to zero.

$$\frac{d \sum \epsilon^2}{d\alpha} = -2 \sum (y - \alpha - \beta x)$$
$$= 0, divide\ through\ by -2\ and\ equate\ the\ result\ to\ zero.$$

$$\sum (y - \alpha - \beta x) = 0, \qquad open\ the\ brackets\ \ and\ make\ y\ the\ subject.$$

$\sum y = n\alpha + \beta \sum x.$ ………………………..(a)

$$differentiate\ with\ respect\ to\ \beta\ to\ get\ the\ second\ normal\ equation$$

$$\frac{d \sum \in^2}{d\beta} = -2x \sum (y - \alpha - \beta x) = 0.\ divide\ through\ by -2\ and\ equate\ to\ zero.$$

$multiply\ by\ x\ and\ make\ y\ the\ subject.\ \sum xy = \alpha \sum \alpha + \beta \sum x^2$…………(b)

- Expressions a and b are normal regression equations that can be solved to determine the parameters α and β.
- Solving expression a, divide through by n and make α the subject.

$\alpha = \bar{y} - \beta \bar{x}$……………………………………..(c)

$solving\ expressions\ a\ and\ b\ b\ using\ cramer's\ rule.$ Write the two expressions in matrix format.

$$\begin{bmatrix} n & \sum x \\ \sum x & \sum x^2 \end{bmatrix} * \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \sum y \\ \sum xy \end{bmatrix}$$

Solve for α and β

$\alpha = \frac{\sum x^2 \sum y - \sum x \sum xy}{n\sum x^2 - (\sum x)^2}$ and $\beta = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - (\sum x)^2}$ or $\beta = \frac{\sum(X-\bar{X})(Y-\bar{Y})}{\sum(X-\overline{X})^2}$ or $\beta = \frac{\sum xy}{\sum x^2}$ as the least squares estimators.

EXAMPLE I: show that LSE are unbiased.

SOLUTION

Required to show that the estimators $\widehat{\beta}$ *and* $\hat{\alpha}$ are unbiased for the parameters β *and* α in the model $y = \alpha + \beta x_i + \varepsilon_{i.}$ that is ; $E(\hat{\beta}) = \beta$ *and* $E(\hat{\alpha}) = \alpha$.

a) Show that $E(\hat{\alpha}) = \alpha$. From the expression $\hat{\alpha} = \bar{Y} - \beta\bar{X}$, introduce expectations on either side $\widehat{E(\alpha)} = E(\ \bar{Y} - \beta\bar{X}\ ).$ *substitute for* $\bar{Y} = \alpha + \beta\bar{X} + \bar{\in}$.
   - $\widehat{E(\alpha)} = E(\ \bar{Y} - \beta\bar{X}\ ) = E(\alpha + \beta\bar{X} + \bar{\in} - \beta\bar{X}) = E(\alpha + \bar{\in})$, *but from the classical assumptions* $E(\in) = 0$ *and the expectation of a constant is a constant.*
   - therefore; $E(\hat{\alpha}) = \alpha$.

b) Show that $E(\hat{\beta}) = \beta$
   - Using $\beta = \frac{\sum xy}{\sum x^2}$, *substitute f for y using* $y = \alpha + \beta x + e$, *where lower case leters refer to a deviation from the mean.* x= X minus mean of X and e is the error minus mean of the errors.
   - Introduce expectations on either side $E(\beta) = E\left(\frac{\sum x(\alpha+\beta x+e)}{\sum x^2}\right) = E(\frac{\alpha\sum x + \beta\sum x^2 + \sum xe}{\sum x^2})$, any deviations from the mean is always equal to zero $\sum x = 0$ *and* $E(xe) = 0$.
   - $E(\hat{\beta}) = E\left(\frac{\beta\sum x^2}{\sum x^2}\right) = E(\beta)$, *but the expectation or mean of a constant is a constant.*
   - *therefore;* $E(\hat{\beta}) = \beta$. this implies unbiasedness.

MEAN AND VARIANCE OF THE PARAMETERS α AND β

The mean for $\hat{\alpha} = E(\hat{\alpha}) = \alpha$ and the variance $v(\hat{\alpha}) = v(\bar{y} - \hat{\beta}\bar{x})$ *this is from* $\hat{\alpha} = \bar{y} - \hat{\beta}\bar{x}$. Therefore; $v(\hat{\alpha}) = v(\bar{y}) + \bar{x}^2 v(\hat{\beta})$. substituting for v(Y) gives $v(\hat{\alpha}) = \delta^2[\frac{1}{n} + \frac{\bar{X}^2}{\sum(X-\bar{X})^2}]$

The mean for $\hat{\beta} = E(\hat{\beta}) = \beta$ and the variance v($\hat{\beta}$) $= \frac{\delta^2}{\sum(x_i - \bar{x})^2}$ where $\delta^2 = \frac{\sum e_i^2}{n-p} = \sum \frac{(y_i - \hat{y}_i)^2}{n-p}$. OR $\delta^2 = \frac{\sum y^2 - \alpha \sum y - \beta \sum xy}{n-p}$, where p are the parameters to be estimated. For simple regression model where only α and β are estimated, p =2.

EXAMPLE: USING THE DATA BELOW, FIT A REGRESSION EQUATION TO THE DATA, DETERMINE THE MEAN AND VARIANCE OF THE ESTIMATED PARAMETERS.

X	77	50	71	72	81	94	96	99	67
Y	82	66	78	34	47	85	99	99	98