Study Unit 3: Electric Potential

Introduction

The first half of this Unit deals mainly with the potential associated with an electric field. The second half covers a number of mathematical topics that will be critical in our treatment of electromagnetism. The potential difference between two points is defined to be the negative line integral of the electric field. Equivalently, the electric field equals the negative gradient of the potential. Just as the electric field is the force per unit charge, the potential is the potential energy per unit charge. This unit involves a number of examples involving the calculation of the potential due to a given charge distribution, this unit also describes more bout electric potential. One important example is the dipole, which consists of two equal and opposite charges. Turning to mathematics, we introduce the divergence, which gives a measure of the flux of a vector field out of a small volume. We prove Gauss's theorem (or the divergence theorem) and then use it to write Gauss's law in differential form.

Learning Outcomes of Study Unit 3

Upon completion of this study unit, you should be able

3.1 Define electric potential and electric potential energy

3.2 State the expression for electric potential at appoint in electric field of point charge

3.3 Calculate the electric potential due to

- (i) Linear charge distribution of charge density
- (ii) Volume charge distribution of charge density
- (iii) Surface charge distribution of charge density

3.4 Derive the relationship between electric field intensity and electric potential

3.5 Define equipotential surfaces and its properties

3.6 Define an electric dipole

3.7 Derive an expression for the electric potential of electric dipole

3.1 Electrical Potential Energy

When a test charge is placed in an electric field, it experiences a force.$F_e = q_0\vec{E}$ The work done within the charge-field system by the electric field on the charge through an infinitesimal displacement $\vec{r}$ is $\vec{F}d\vec{r} = q_0\vec{E}.d\vec{r}$

As this work is done by the field, the change in potential energy is $\Delta U = q_0\vec{E}.d\vec{r}$

$$\Delta U = W$$

For a finite displacement of the charge from A to B

$$\Delta U = U_B - U_A = -q_0 \int_A \vec{E}.d\vec{r}$$

Electric Potential is the potential energy per unit charge i.e. $V = \frac{U}{q_o}$

Units of the electric potential is volt (V)

Electric potential (V) is a scalar quantity

As a negative charged particle moves in an electric field

$$\Delta V = \frac{\Delta U}{q_o} = -\int_A^B \vec{E}.d\vec{r}$$

The equations for electric potential between two points A and B can be simplified if the electric field is uniform:

$$V_B - V_A = \Delta V = -\int_A^B \vec{E}.d\vec{r} = -\vec{E}\int_A^B d\vec{r} = -\vec{E}r$$

Similarly, for the positive charge

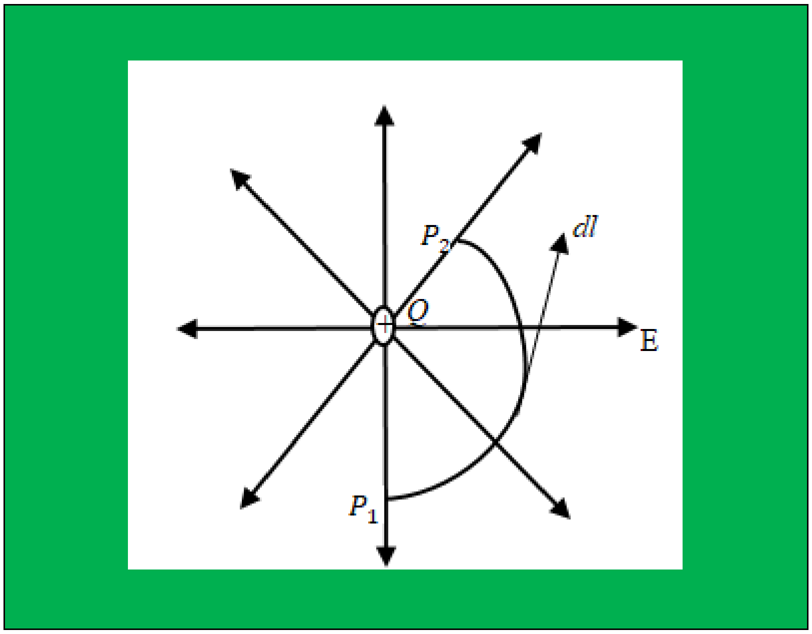
$$V_B - V_A = \Delta V = \int_A^B \vec{E}.d\vec{r} = \vec{E}\int_A^B d\vec{r} = \vec{E}r$$

(i) A positive charge gains electrical potential energy when it is moved in a direction opposite the electric field.

(ii) A negative charge loses electrical potential energy when it is moved in a direction opposite the electric field.

3.2 Electric Potential of Point Charges.

Consider a point charge of 1C moved from point P_1 $to\ point$ P_2 in the electric field of a point charge Q in the free space.



The work done is

$$V_2 - V_1 = \int_{P_1}^{P_2} \vec{E}.dl$$

$$\text{But } \vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$$

$$V_2 - V_1 = \int_{r_1}^{r_2} \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}dl$$

$$= \frac{Q}{4\pi\varepsilon_0}\int_{r_1}^{r_2}\frac{dr}{r^2}$$

$$= \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_1}-\frac{1}{r_2}\right]$$

To obtain electric potential at P_2 we set $V_1 = 0$ at $r_1 = \infty$

Hence $V_2 = \frac{Q}{4\pi\varepsilon_0 r_2}$

Thus, electric potential at a point a distance r from a charge Q in a free space is

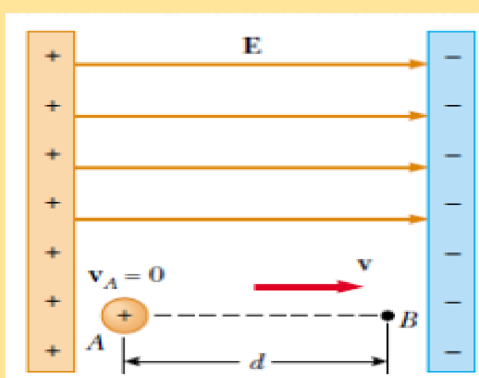
$$V(r) = \frac{Q}{4\pi\varepsilon_0 r}$$

The total potential at some point P due to several point charges is the algebraic sum of the potentials due to the individual charges

$$V_{Total} = \sum_{i}^{N} V_i = k\sum_{i}^{N}\frac{Q_i}{r_i}$$

Once again, this called superposition principle.

Example 1. A proton is released from rest in a uniform electric field of $8.0x10^4 V/m$. The proton under goes a displacement of 0.50m in the direction of E. (i) Find the change in electric potential between points A and B. (ii) Find the change in potential energy of the proton–field system for this displacement.



Solution

(i) Because the positively charged proton moves in the direction of the field, we expect it to move to a position of lower electric potential

$$\Delta V = -\vec{E}d$$

$$\Delta V = -8.0x10^4x0.5$$

$$\Delta V = -4.0x10^4V$$

(ii)

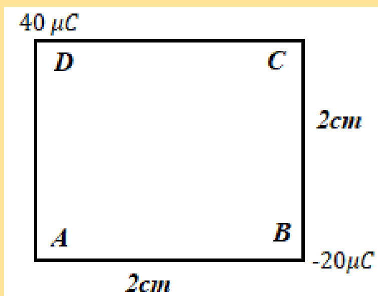
$$\Delta U = q_0.\Delta V = e\Delta V$$

$$\Delta U = -1.6x10^{-19}x\ 4.0x10^4$$

$$\Delta U = -6.4.0x10^{-15}J$$

Example

The figure below shows point charges 40μC and -20μC placed at the corners of a square of side 0.5m as shown below. Calculate; (i) Electric potential at C (ii) Potential energy of 10μC charge placed at C



Solution

(i) Electric potential at C

Using

$$V(r) = \frac{Q}{4\pi\varepsilon_0 r}$$

$$V_D = \frac{9.0x10^{-9}40.0x10^{-6}}{2.0x10^{-2}}$$

$$V_D = 1.8x10^7 V$$

$$V_B = \frac{9.0x10^{-9} 20.0x10^{-6}}{2.0x10^{-2}}$$

$$V_B = 9.0x10^6 V$$

$$V = V_A + V_B$$

$$V = 9.0x10^7 + 1.8x10^7 = 2.7x10^7 V$$

(ii) Potential energy of $10\mu C$ charge placed at C

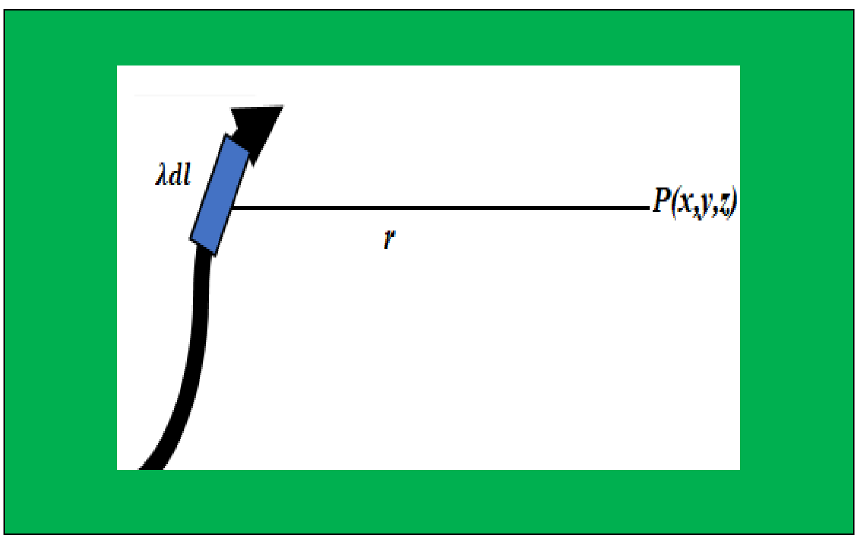
$$P.E = Q.V$$

$$P.E = 10x10^{-6} x 2.7x10^7$$

$$P.E = 270J$$

3.3 Electric potential of continuous charge distribution

3.3.1 Consider Linear charge distribution of charge density ($\lambda C m^{-1}$)



$$dV_p = k\frac{Q}{r}$$

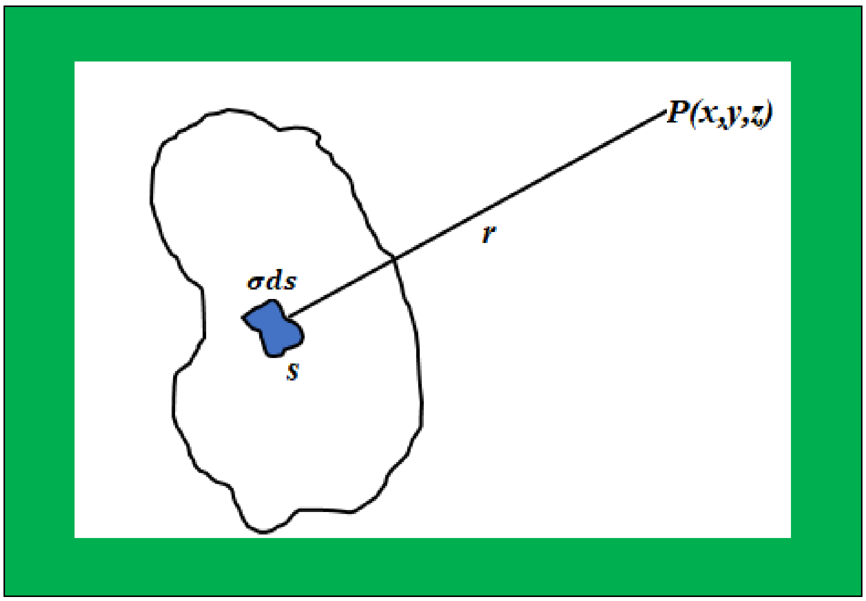
$$dV_p = k\frac{\lambda dl}{r}$$

Integrating both sides we obtain

$$V_p = \int_c k\frac{\lambda dl}{r}$$

3.3.2 Surface charge distribution of charge density (σCm^{-2})

$$dq = \boldsymbol{\sigma ds}$$



The electric potential on the small charge element dq at point P is

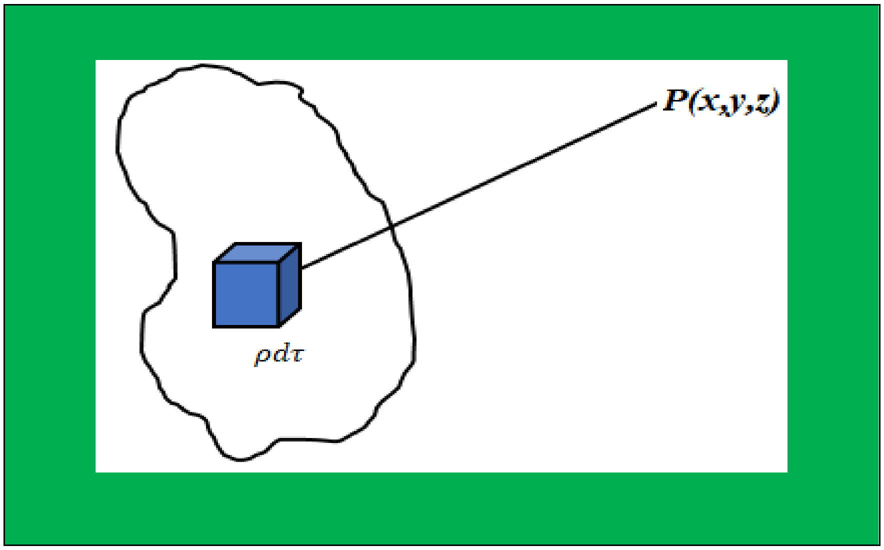
$$d\vec{V}_p = k\frac{\sigma ds}{r} \qquad \text{Where s is the surface}$$

Integrating both sides

$$\vec{V}_p = \int_s k\frac{\sigma ds}{r}$$

3.3.3 Volume charge distribution of charge density(ρCm^{-3}),

$$dq = \rho d\tau$$



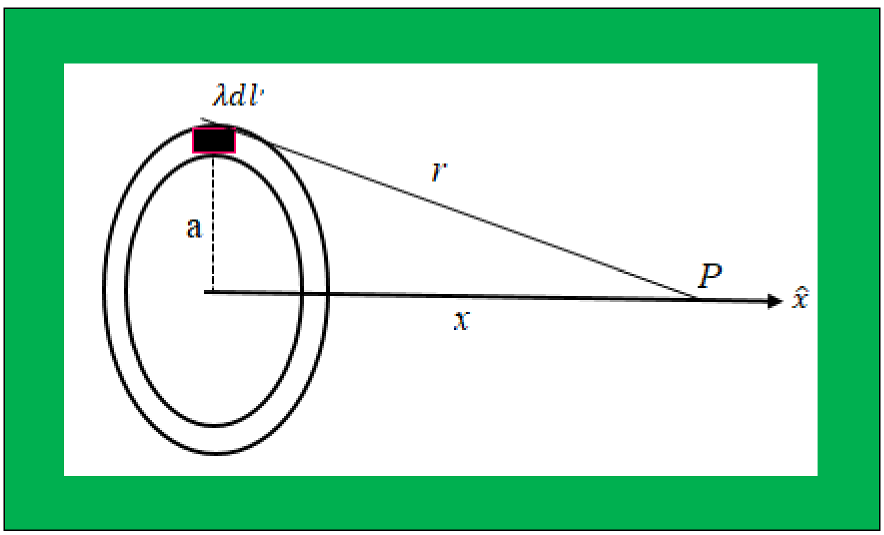
The electric field on the small charge element dq at point P is $d\vec{V}_p = k\frac{\rho d\tau}{r}$

Integrating both sides

$$V_p = \int_\tau k\frac{\rho d\tau}{r}$$

Example I

Find an expression for electric potential at a point P on the axis of a uniformly charged ring of radius a



$$dV_p = k\frac{dQ}{r}$$

$$dV_P = \frac{\lambda dl'}{r}$$

Integrating both sides

$$V_P = k\int_0^{2\pi a}\frac{\lambda dl'}{(x^2+a^2)^{\frac{1}{2}}}$$

$$V_P = k\frac{\lambda}{(x^2+a^2)^{\frac{1}{2}}}\int_0^{2\pi a} dl'$$

$$V_P = \frac{k\lambda(2\pi a)}{(x^2+a^2)^{\frac{1}{2}}}$$

But $\lambda = \frac{Q}{2\pi a}$

Hence

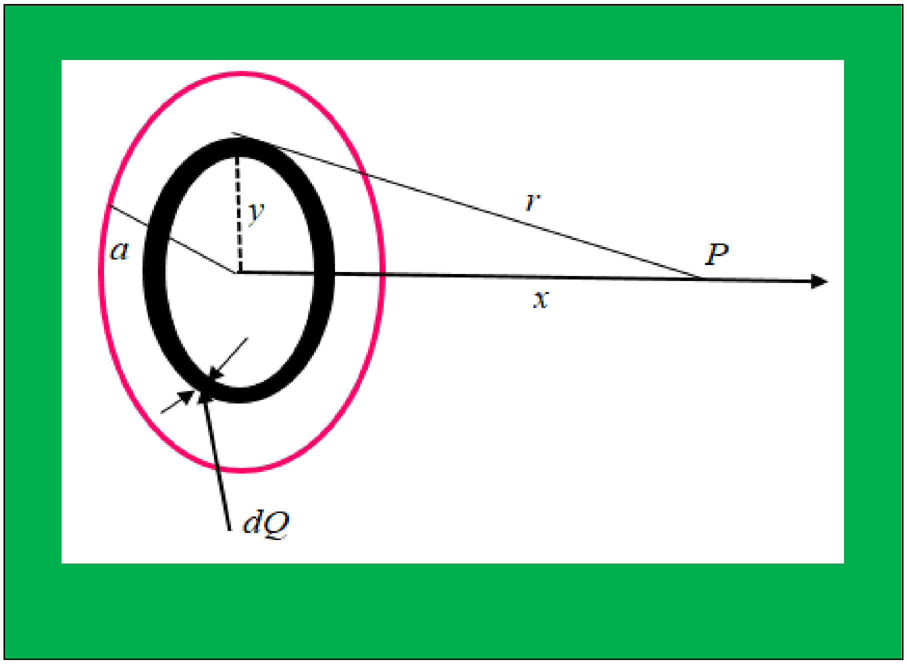
$$V_P = k\frac{Q}{(x^2+a^2)^{\frac{1}{2}}}$$

$$V(x \gg a) \to \frac{kQ}{x}$$

There fore as x increases the ring acts as the point charge

Example II

Find electric potential at a point on the axis of a uniformly charged disk whose surface of the charge is σCm^{-2}



$$dV_p = k\frac{dQ}{r}$$

$$dV_P = \frac{\sigma(2\pi y dy)}{r}$$

Integrating both sides

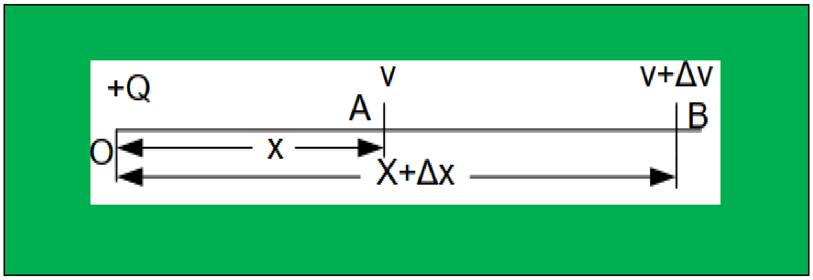
$$V_P = k\int_0^a \frac{\sigma 2\pi y dy}{(x^2 + a^2)^{\frac{1}{2}}}$$

$$V_P = 2\pi\sigma k\int_0^a \frac{y dy}{(x^2 + a^2)^{\frac{1}{2}}}$$

$$V_P = \frac{\sigma}{2\varepsilon_0}\left[(x^2 + a^2)^{\frac{1}{2}} - x\right]$$

3.4 Electric Potential Gradient (Relation Between E And V)

Consider two points A and B in an electric field which are $\Delta x m$ apart



if the potential at A is v and that at B is $v + \Delta v$. Then potential difference between A and B is

$$\boldsymbol{V_{AB} = V_B - V_A}$$

$$\boldsymbol{V_{AB} = V - (V + \Delta V)}$$

$$\boldsymbol{V_{AB} = -\Delta V} \qquad \textbf{(1)}$$

Work done to move 1C of change from A to B is equal to p.d and is given by

$$\boldsymbol{V_{AB} = E\Delta x} \qquad \textbf{(2)}$$

Considering equation 1 and 2

$$\boldsymbol{E\Delta x = -\Delta V}$$

$$E = -\frac{\Delta V}{\Delta x}$$

3.5 Equipotential Surfaces

An equipotential surface is any two-dimensional surface over which the electric potential is constant and work done moving charge from one point on surface to another is zero

The direction of force is always at right angles to equipotential surfaces. This implies that the is no component of electric field inside the surface

3.5.1 Properties of equipotential surface

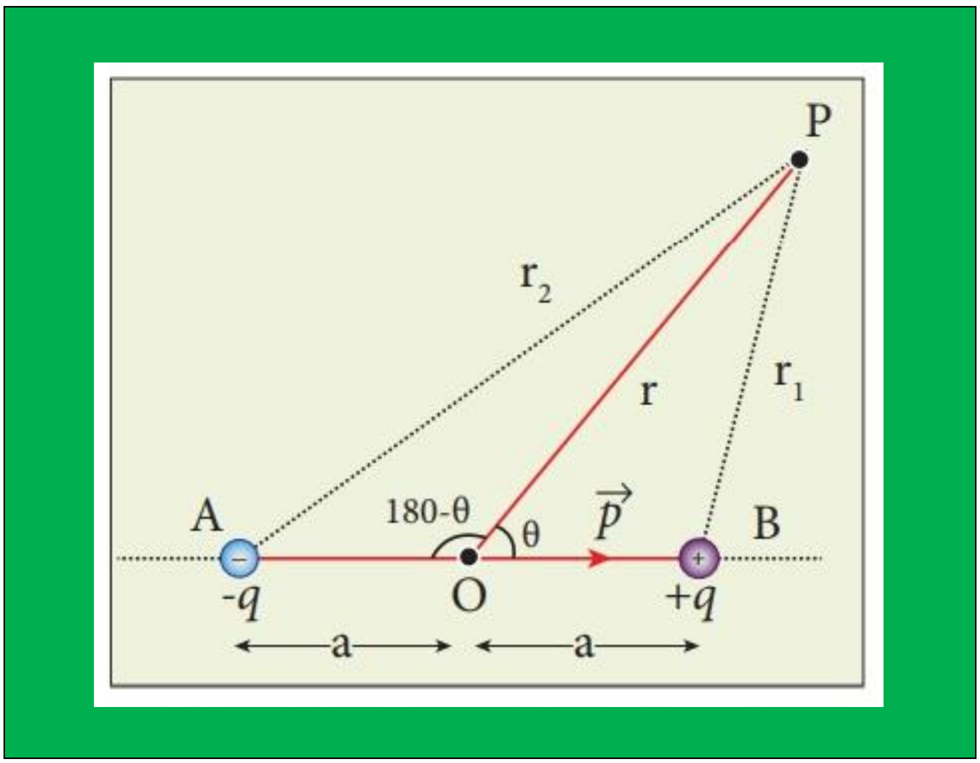
(i) Work done along an equipotential surface is zero

(ii) Electric field intensity along surfaces is zero

(iii)The surfaces are at right angles to the line of force

3.6 Electric dipole

An electric dipole consists of two unlike charges of equal magnitude separated by a distance much smaller than the distances to points at which electric field of the dipole is being investigated



Let r_1 be the distance of point P from +q and r_2 be the distance of point P from –q.

Potential at P due to charge

$$+q = \frac{1}{4\pi\varepsilon_0}\frac{q}{r_1}$$

Potential at P due to charge

$$-q = -\frac{1}{4\pi\varepsilon_0}\frac{q}{r_1}$$

Total potential at the point P,

$$V = \frac{1}{4\pi\varepsilon_0} q\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \qquad (1)$$

Suppose if the point P is far away from the dipole, such that r>>a, then equation (1) can be expressed in terms of r.

By the cosine law for triangle BOP,

$${r_1}^2 = r^2 + a^2 - 2racos\theta$$

$${r_1}^2 = r^2\left(1 + \frac{a^2}{r^2} - \frac{2a}{r}cos\theta\right)$$

Since the point P is very far from dipole, then r>>a. As a result the term a^2/r^2 is very small and can be neglected. Therefore

$${r_1}^2 = r^2\left(1 - 2a\frac{cos\theta}{r}\right)$$

$$r_1 = r\left(1 - 2a\frac{cos\theta}{r}\right)^{\frac{1}{2}}$$

$$\frac{1}{r_1} = \frac{1}{r}\left(1 - \frac{2a}{r}cos\theta\right)^{-\frac{1}{2}}$$

Since $a/r \ll 1$, we can use binomial theorem and retain the terms up to first order

$$\frac{1}{r_1} = \frac{1}{r}\left(1 + \frac{a}{r}cos\theta\right) \qquad (2)$$

Similarly applying the cosine law for triangle AOP,

$$r_2^2 = r^2 + a^2 - 2ra\cos(180 - \theta)$$

since $\cos(180-\theta) = -\cos\theta$ we get

$$r_2^2 = r^2 + a^2 + 2ra\cos\theta$$

Neglecting the term $\frac{a^2}{r^2}$ (because r>>a)

$$r_2^2 = r^2\left(1 + \frac{2a\cos\theta}{r}\right)$$

$$r_2 = r\left(1 + \frac{2a\cos\theta}{r}\right)^{\frac{1}{2}}$$

Using Binomial theorem, we get

$$\frac{1}{r_2} = \frac{1}{r}\left(1 - a\frac{cos\theta}{r}\right) \qquad (3)$$

Substituting equation (3) and (2) in equation (1),

$$V = \frac{1}{4\pi\varepsilon_0}q\left[\frac{1}{r}\left(1 + a\frac{cos\theta}{r}\right) - \frac{1}{r}\left(1 - a\frac{cos\theta}{r}\right)\right]$$

$$V = \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{r}\left(1 + a\frac{cos\theta}{r} + 1 + a\frac{cos\theta}{r}\right)\right]$$

$$V = \frac{1}{4\pi\varepsilon_0}\frac{2aq}{r^2}cos\theta$$

But the electric dipole moment $p = 2qa$ and we get,

$$V = \frac{1}{4\pi\varepsilon_0}\left(\frac{p\cos\theta}{r^2}\right)$$

Now we can write p $cos\theta = \vec{p}x\hat{r}$ where r is the unit vector from the point O to point P. Hence the electric potential at a point P due to an electric dipole is given by

$$V = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}.\hat{r}}{r^2} \qquad (4)$$

Equation (1.38) is valid for distances very large compared to the size of the dipole. But for a point dipole, the equation (1.38) is valid for any distance.

Special cases

Case (i) If the point P lies on the axial line of the dipole on the side of +q, then θ = 0. Then the electric potential becomes

$$V = \frac{1}{4\pi\varepsilon_0}\frac{p}{r^2} \qquad (5)$$

Case (ii) If the point P lies on the axial line of the dipole on the side of –q, then θ = 180°, then

$$= \frac{1}{4\pi\varepsilon_0}\frac{p}{r^2} \qquad (6)$$

Case (iii) If the point P lies on the equatorial line of the dipole, then θ = 90°. Hence

$$V = 0 \qquad (7)$$

Electric field intensity at P

$$\vec{E} = -\nabla V$$

In polar coordinates $\nabla = \hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta}$

But $V = \frac{pcos\theta}{4\pi\varepsilon_0 r^2}$

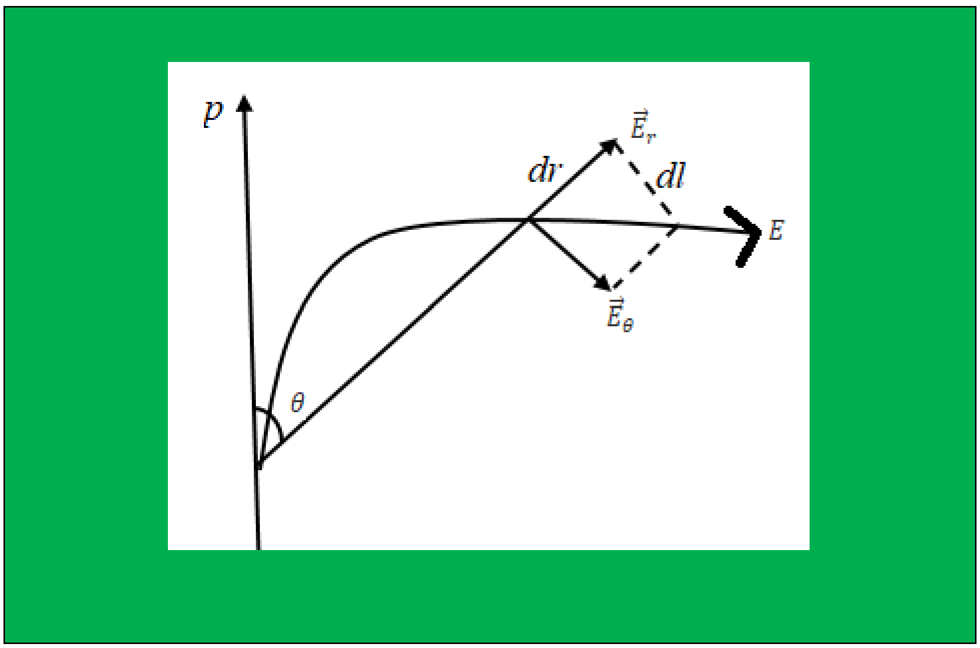
$$\vec{E} = -\frac{\partial V}{\partial r}\hat{r} - \frac{\hat{\theta}}{r}\frac{\partial V}{\partial \theta}$$

$$\vec{E}_r = \frac{\partial V}{\partial r} = \frac{2pcos\theta}{4\pi\varepsilon_0 r^3}$$

$$\vec{E}_\theta = \frac{\hat{\theta}}{r}\frac{\partial V}{\partial \theta} = \frac{pcos\theta}{4\pi\varepsilon_0 r^3}$$

$$E = \frac{2pcos\theta}{4\pi\varepsilon_0 r^3}\,\hat{r} + \frac{pcos\theta}{4\pi\varepsilon_0 r^3}$$

Electric field lines of an electric dipole



Since E is parallel to the path element dl and $dl = (dr)\hat{r} + (rd\theta)\hat{\theta}$

$$E = E_r\hat{r} + E_\theta\hat{\theta}$$

We have $\frac{rd\theta}{E_\theta} = \frac{dr}{E_r}$

Or $\frac{dr}{r} = \frac{E_r}{E_\theta}d\theta = \frac{2cos\theta d\theta}{sin\theta}$

Integrating both sides we get

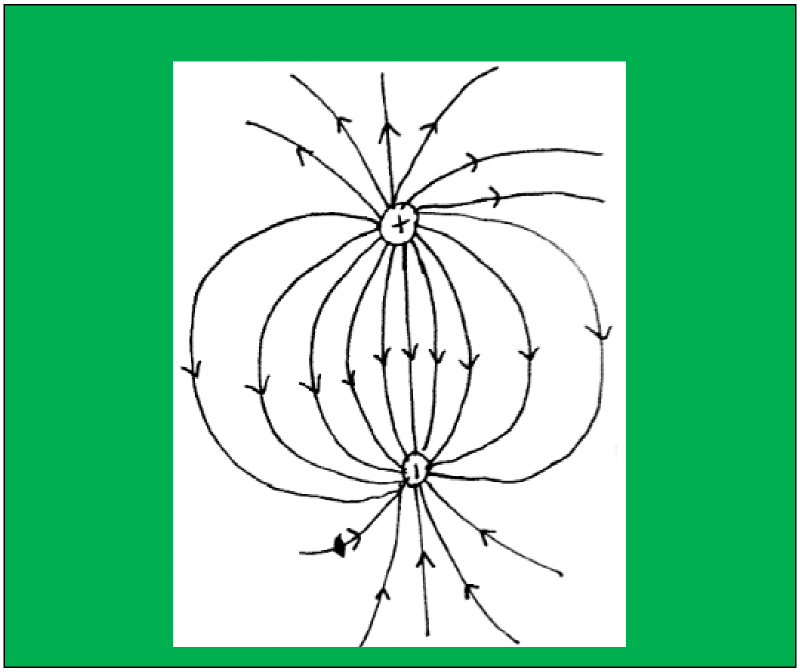
$$lnr = 2lnsin\theta + constant$$

Or $ln\left[\frac{r}{sin^2\theta}\right] = constant$

Hence $r = Asin^2\theta$

Where A is a constant parameter which takes on different values of from one electric field line to another

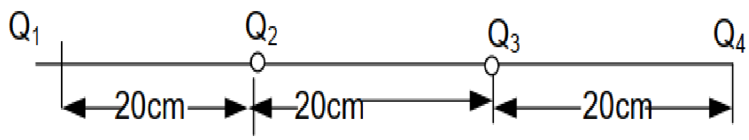
The sketch of electric field lines of an electric dipole is shown below



Self-Review Question (SRQs) for study unit 3

Now that you have completed this unit, you can measure how well you achieved its learning outcomes by answering the following questions. You can check your answers with the Notes on Self-Review Questions at the end of this study unit

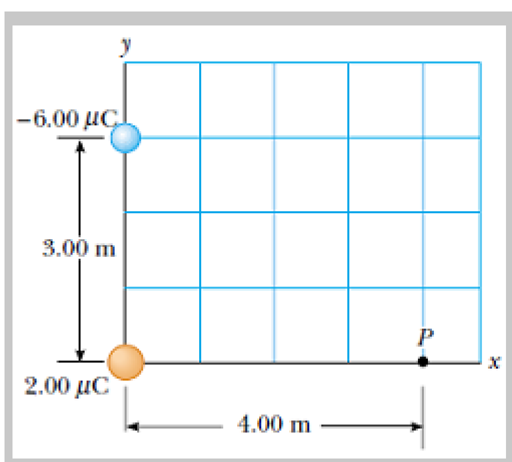
1. The figure shows charges $Q1$ $Q2$ $Q3$ and $Q4$ of -2μC, 3μC, -5μC and 8μC are arranged on a straight line in vacuum



(a) Calculate potential energy at Q2

(b) what is the significance of the sign of the potential energy above

2. An amount of charge of 12μC is uniformly distributed on metal ring of radius 8.0cm. find the electric potential at appoint on the axis of the ring a distance of 10.0cmfrom the Centre of the ring
3. A conducting sphere of radius 9.0 cm is maintained at an electric potential of 10kV. Calculate the charge on the sphere.
4. Derive an expression for the electric potential at a point of a distance r, from a fixed charge.
5. A charge$q_1 = 2.00\mu C$ is located at the origin, and a charge$q_2 = 6.00\mu C$ is located at (0,3.00) m, as shown in Figure



(i) Find the total electric potential due to these charges at the point P, whose coordinates are (4.00,0) m.

(ii) Find the electric force between the two charges

(iii) Find the potential energy of the system

6. Define equipotential surfaces and state its properties
7. The electric dipole moment of a $H^{+}Cl^{-}$ molecule is $3.4x10^{-30}Cm$. Calculate the magnitude of the torque that electric field of intensity $2.0x10^{6}NC^{-1}$exerts on this molecule when the axis of the molecule when the axis of the molecule makes an angle of 30^{0} with the electric field

Summary

- The work done in taking a unit positive charge from one point to another in electric field is independent of the path chosen between the two points
- The potential at a point is the work done in carrying a unit positive charge from infinity to the point against the electric field.
- The potential V at a distance r from a point charge q is given by $V(r) = \frac{Q}{4\pi\varepsilon_0 r}$
- The potential difference V_{AB} between two-point B and A is equal to the work done in taking a unit positive charge from A to B. if a charge q is taken from A to B, then the work done is.$V_{AB} = V_a - V_B$
- The unit of potential difference is the volt. The potential difference between point A and B is 1 volt when the work done in carrying unit positive charge between the two points is equal to 1 joule.
- Electric potential of continues distribution of charge is also discussed
- Electric field intensity is given by $E = \frac{2pcos\theta}{4\pi\varepsilon_0 r^3}\,\hat{r} + \frac{pcos\theta}{4\pi\varepsilon_0 r^3}$

References

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